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A note on helicity conservation in steady fluid flows

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A theorem on helicity conservation proved by Moffatt (1969) for the flows of inviscid barotropic fluids is generalized, for steady flows, to any fluid in which vorticity field lines are material. To make this generalization, the helicity within a volume V enclosed by a material surface S must be defined by the volume integral,

$$\mathscr{H}'_S(t) \equiv \int_V (\lambda/J)\, \boldsymbol{m}\cdot\boldsymbol{v}\, \mathrm{d}V,$$

where $\boldsymbol{v}$ is the fluid velocity, $\boldsymbol{m}$ is a unit vector tangent to a vorticity line, λ is the vorticity line stretch (Casey & Naghdi 1991), and J is the determinant of the deformation gradient tensor. For the case of an inviscid barotropic fluid, $\mathscr{H}'_S$ differs only by a constant factor from the helicity integral defined originally by Moffatt (1969). The condition under which $\mathscr{H}'_S$ is invariant under steady fluid motion is also the condition necessary and sufficient for the existence of a permanent system of surfaces on which both the stream lines and the vorticity lines lie (Sposito 1997). These surfaces and the helicity invariant $\mathscr{H}'_S$ figure importantly in the topological classification of integrable steady fluid flows, including flows with dissipation, in which vorticity lines are material.

1. Introduction

The helicity of a fluid flow with velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ and vorticity field $\boldsymbol{\omega}(\boldsymbol{x}, t) = \boldsymbol{\nabla} \times \boldsymbol{v}(\boldsymbol{x}, t)$ is defined by the volume integral (Moffatt 1969)

$$\mathscr{H}_S(t) \equiv \int_V \boldsymbol{\omega}\cdot\boldsymbol{v}\, \mathrm{d}V, \tag{1.1}$$

where S is any orientable material surface enclosing fluid within the volume V. The element $\boldsymbol{\omega}\cdot\boldsymbol{v}\,\mathrm{d}V$ of the integral in (1.1) reflects local helical motion in the fluid. If S is a closed vorticity surface (i.e. $\boldsymbol{\omega}\cdot\boldsymbol{n} = 0$ on the oriented surface S, where $\boldsymbol{n}$ is a unit vector along an outward normal to S), and if the fluid is inviscid, barotropic, and acted upon by conservative body forces, Moreau (1961) and, independently, Moffatt (1969) have shown that $\mathscr{H}_S(t)$ is invariant under the fluid motion. The physical and topological implications of helicity conservation have been discussed extensively (Moffatt 1969, 1983, 1985, 1990; Moffatt & Tsinober 1992; Moffatt & Ricca 1992; Ricca & Berger 1996). Mobbs (1981), Gaffet (1985), and most recently Salmon (1988) have proved helicity conservation theorems for the flows of inviscid fluids that are not barotropic, but that admit the Ertel potential vorticity (Ertel 1942; Truesdell 1954) as an invariant of the motion. These generalizations permit $\boldsymbol{v}$ in (1.1) to be replaced by $\boldsymbol{v} - \eta\boldsymbol{\nabla}\mathscr{S}$, where η is a coefficient whose material derivative equals the absolute temperature and $\mathscr{S}$ is specific entropy. The vorticity in (1.1) is correspondingly replaced by $\boldsymbol{\omega}-\nabla\eta\times\nabla\mathscr{S}$, which vector must now be perpendicular to the unit vector $\boldsymbol{n}$ if $\mathscr{H}_S(t)$ is to be an invariant of the motion. Salmon (1988) has stressed the close connection between this type of generalized helicity conservation and the invariance of fluid density and entropy under relabelling of the Lagrangian coordinates assigned to fluid particles.

The demonstration that $\mathscr{H}_S(t)$ is constant, if S is a closed vorticity surface, usually involves consideration of the fluid equation of motion (Gaffet 1985; Moffatt & Tsinober 1992), but this approach is not necessary. For any fluid motion to be circulation-preserving, the purely kinematic condition that the curl of the fluid acceleration vanish,

$$\nabla\times\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}=\frac{\partial\boldsymbol{\omega}}{\partial t}+\nabla\times(\boldsymbol{\omega}\times\boldsymbol{v})=\mathbf{0}, \tag{1.2}$$

is both necessary and sufficient (Truesdell 1954). Under (1.2), it follows that

$$\begin{aligned}\frac{\mathrm{d}\mathscr{H}_S}{\mathrm{d}t}&=\int_V\left[\frac{\mathrm{D}(\boldsymbol{\omega}\cdot\boldsymbol{v})}{\mathrm{D}t}+\boldsymbol{\omega}\cdot\boldsymbol{v})\right]\mathrm{d}V\\&=\int_V\left(\boldsymbol{\omega}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{\omega}\cdot\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}\right)\mathrm{d}V\\&=\int_V\nabla\cdot(\boldsymbol{\omega}Q)\,\mathrm{d}V=\int_S\mathrm{Q}(\boldsymbol{x},t)\,\boldsymbol{\omega}\cdot\boldsymbol{n}\,\mathrm{d}S,\end{aligned} \tag{1.3}$$

where two vector identities,

$$\nabla\times(\boldsymbol{f}\times\boldsymbol{g})=\boldsymbol{g}\cdot\nabla\boldsymbol{f}-\boldsymbol{f}\cdot\nabla\boldsymbol{g}-\boldsymbol{g}\nabla\cdot\boldsymbol{f}+\boldsymbol{f}\nabla\cdot\boldsymbol{g}, \tag{1.4}$$

$$\boldsymbol{f}\cdot\nabla\boldsymbol{f}=\tfrac{1}{2}\nabla|\boldsymbol{f}|^2+(\nabla\times\boldsymbol{f})\times\boldsymbol{f}, \tag{1.5}$$

have been applied ($\boldsymbol{f}(\boldsymbol{x},t)$ and $\boldsymbol{g}(\boldsymbol{x},t)$ being continuously differentiable vector fields) along with an implication of (1.2),

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}=\nabla F, \tag{1.6}$$

where $F(\boldsymbol{x},t)$ is a continuously differentiable scalar field, and the scalar field $Q(\boldsymbol{x},t)$ is defined by

$$Q(\boldsymbol{x},t)\equiv\tfrac{1}{2}|\boldsymbol{v}|^2+F(\boldsymbol{x},t). \tag{1.7}$$

The right-hand side of (1.3) then vanishes if S is a closed vorticity surface.

As pointed out by Moffatt & Tsinober (1992), for the vorticity lines of a fluid flow governed by (1.6) that lie on a family of nested surfaces, there is a corresponding family of helicity invariants. When the fluid flow is steady, it follows from (1.5) and (1.6) that vorticity lines should lie on the level surfaces of a scalar function $H(\boldsymbol{x})$ whose gradient is defined by (Moffatt 1969)

$$\nabla H\equiv\boldsymbol{v}\times\boldsymbol{\omega}\quad(\boldsymbol{v}\times\boldsymbol{\omega}\neq\mathbf{0}). \tag{1.8}$$

Arnol'd (1965, 1989) and Kozlov (1984), using topological methods, have shown that (1.8) does indeed lead to the existence of families of closed vorticity surfaces that are (diffeomorphic to) tori, if the domain of steady flow under (1.6) is oriented, connected, and compact, and if stagnation points of $\boldsymbol{\omega}$ and $\boldsymbol{v}$ are avoided (finite in number because of smoothness and compactness conditions). This broad result applies to any smooth, circulation-preserving, steady velocity field in a compact three-dimensional domain, that of an inviscid barotropic fluid being just one physical example.

The purpose of this note is to show that the Moffatt helicity conservation theorem can be generalized to cover a broader class of fluid flows than those to which (1.2) applies. This class of fluid flows is that for which the right-hand side of (1.2) is equal not to $\mathbf{0}$, but instead is proportional to the vorticity field $\boldsymbol{\omega}(\boldsymbol{x})$. This condition, unlike (1.2), is consistent with the presence of non-conservative forces acting on a fluid (Batchelor 1967), but implies that the field lines of $\boldsymbol{\omega}(\boldsymbol{x})$ remain governed by purely advective processes, despite the existence of non-conservative forces and the vorticity diffusion they engender (Truesdell 1954). It is this very special kinematic circumstance that makes possible an extension of the Moffatt helicity conservation theorem.

2. Theorem on helicity conservation

When the (1.2) holds, a steady vorticity field in an incompressible fluid changes with time in the same manner as does a material line element in the fluid (Batchelor 1967), a result that follows directly from the kinematic condition imposed in (1.6) (Casey 1996). When (1.6) does not hold and the fluid considered is compressible, but with the right-hand side of (1.2) proportional to the vorticity field, the field lines of $\boldsymbol{\omega}(\boldsymbol{x})$ will remain material (even though the vector field $\boldsymbol{\omega}(\boldsymbol{x})$ is not) because of the Helmholtz–Zoraski criterion (Truesdell 1954). In this latter situation, it is $\lambda \boldsymbol{m}$, instead of $\boldsymbol{\omega}(\boldsymbol{x})$, that is a material vector field, $\boldsymbol{m}$ being a unit vector tangent to a vorticity line of $\boldsymbol{\omega}$ and λ the vorticity line stretch, defined through the equation

$$\lambda \boldsymbol{m} = \boldsymbol{F}\boldsymbol{M}, \tag{2.1}$$

where $\boldsymbol{F}$ is the deformation gradient tensor that connects a material line element in a Lagrangian reference configuration to its counterpart in the present configuration (Ottino 1990; Casey & Naghdi 1991). Thus $\boldsymbol{M}$ is a constant unit vector tangent to a vorticity line in a reference configuration and $\lambda \boldsymbol{m}$ is its present, stretched counterpart. The formal definition of λ follows from (2.1) as

$$\lambda = (\boldsymbol{M}\cdot \boldsymbol{C}\boldsymbol{M})^{1/2}, \tag{2.2}$$

where $\boldsymbol{C} \equiv \boldsymbol{F}^T\boldsymbol{F}$ is the (symmetric) right Cauchy–Green tensor (Marsden & Hughes 1994).

Equation (2.1) shows that the material derivative of the vector field $\lambda \boldsymbol{m}$ derives solely from that of the tensor $\boldsymbol{F}$:

$$\frac{\mathrm{D}}{\mathrm{D}t}(\lambda \boldsymbol{m}) = \frac{\mathrm{D}\boldsymbol{F}}{\mathrm{D}t}\boldsymbol{M} = \lambda \boldsymbol{m}\cdot\nabla \boldsymbol{v}, \tag{2.3}$$

where a standard result for the material derivative of $\boldsymbol{F}$ has been applied. Equation (2.3) is a conventional mathematical representation for the rate of change of a material line element in a fluid (Batchelor 1967). For a steady fluid flow, it can be rewritten in a commutator form,

$$\boldsymbol{v}\cdot\nabla(\lambda \boldsymbol{m}) - (\lambda \boldsymbol{m})\cdot\nabla \boldsymbol{v} = \mathbf{0}, \tag{2.4}$$

which implies that the steady rate of movement of $\lambda \boldsymbol{m}$ along a stream line is equal to that of $\boldsymbol{v}$ along a vorticity line (that is, $\lambda \boldsymbol{m}$ and $\boldsymbol{v}$ are commuting vector fields). Therefore, the same point in space can be reached irrespective of the order in which smooth, intersecting stream lines and vorticity lines are traversed to get there. Equation (2.4) is both necessary and sufficient (Sposito 1997) for the existence of families of closed vorticity surfaces that are (diffeomorphic to) tori. (Sufficiency also was demonstrated by Marris 1969.) These smooth tori are covered by intersecting stream lines and vorticity lines, similarly to the case of an inviscid barotropic fluid in (non-

Beltrami) steady flow within a compact three-dimensional domain (Lamb 1878; Arnol'd 1965, 1989; Kozlov 1984). The relevant topological theorem, of which the result here is a special case (Godbillon 1983), is that the only smooth, connected, compact, orientable two-dimensional manifolds which can be spanned by a vector field without stagnation points are the torus (closed surface) and the cylinder (surface with a boundary).

Casey & Naghdi (1991) have derived an explicit form of the Helmholtz–Zoraski criterion that ultimately can be expressed in terms of $\lambda \boldsymbol{m}$ and $J > 0$, the determinant of $\boldsymbol{F}$ (Theorem I(d) in Casey & Naghdi 1991). For steady fluid flows, it reads

$$\nabla \times (\boldsymbol{\omega} \times \boldsymbol{v}) = \frac{\mathrm{D}\omega^0}{\mathrm{D}t}\frac{\boldsymbol{\omega}}{\omega^0}, \tag{2.5}$$

where $\boldsymbol{\omega}^0$ is the Lagrangian vorticity field, defined by the Piola transformation

$$\boldsymbol{\omega}^0 = J\boldsymbol{F}^{-1}\boldsymbol{\omega} \tag{2.6}$$

(Piola transformations are discussed in detail in the textbook by Marsden & Hughes 1994). Casey & Naghdi (1991) investigated the kinematics of $\boldsymbol{\omega}^0$ extensively. They showed that

$$\omega/\omega^0 = \lambda/J \tag{2.7}$$

relates the magnitudes of Eulerian and Lagrangian vorticity vectors to the vorticity line stretch and the determinant of $\boldsymbol{F}$.

From (2.4), (2.5), and (2.7), an explicit generalization of (1.8) can be derived (Sposito 1997):

$$\omega^0 \nabla H = \boldsymbol{\omega} \times \boldsymbol{v}. \tag{2.8}$$

Equation (2.8) requires only that the Lamb vector $\boldsymbol{\omega} \times \boldsymbol{v}$ be integrable (as opposed to irrotational), a consequence of material vorticity lines in a steady fluid flow. The Casey & Naghdi (1991) theorem expressed in (2.5) then yields the explicit integrating factor ω^0 in (2.8). Equation (2.7) can be applied to (2.8) in order to derive a generalization of (1.2) for the case of steady material vorticity lines,

$$\nabla \times [(\lambda/J)\, \boldsymbol{m} \times \boldsymbol{v}] = \boldsymbol{0}. \tag{2.9}$$

In order to recover (1.2), one notes that ω differs from λ/J only by a constant factor if $\boldsymbol{\omega}(\boldsymbol{x})$ is a material vector field (i.e. ω^0 is constant; see Corollary III in Casey & Naghdi 1991).

Comparison of (1.2) and (2.9) indicates that an appropriate generalization of (1.1) is then

$$\mathscr{H}'_S(t) \equiv \int_V (\lambda/J)\, \boldsymbol{m}\cdot\boldsymbol{v}\, \mathrm{d}V. \tag{2.10}$$

With (2.4), the time-dependence of $\mathscr{H}'_S(t)$ can be evaluated:

$$\begin{aligned}\frac{\mathrm{d}\mathscr{H}'_S}{\mathrm{d}t} &= \int_V \left\{\frac{\mathrm{D}[(\lambda/J)\, \boldsymbol{m}\cdot\boldsymbol{v}]}{\mathrm{D}t} + (\lambda/J)\, \boldsymbol{m}\cdot\boldsymbol{v}\, (\nabla\cdot\boldsymbol{v})\right\} \mathrm{d}V \\ &= \int_V (\lambda/J)\, \boldsymbol{m}\cdot\left(\nabla\boldsymbol{v}\cdot\boldsymbol{v} + \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}\right) \mathrm{d}V \\ &= \int_V (\lambda/J)\, \boldsymbol{m}\cdot\nabla v^2\, \mathrm{d}V = \int_V \nabla\cdot[(\lambda/J)\, \boldsymbol{m} v^2]\, \mathrm{d}V \\ &= \int_S (v^2\lambda/J)\, \boldsymbol{m}\cdot\boldsymbol{n}\, \mathrm{d}S, \end{aligned} \tag{2.11}$$

where S is a closed surface surrounding a domain of volume V. The second step in (2.11) makes use of (1.5) and the well-known result

$$\frac{\mathrm{d}J}{\mathrm{d}t} = (\boldsymbol{\nabla}\cdot\boldsymbol{v})\,J, \tag{2.12}$$

while the penultimate step takes advantage of the fact that the vector field $(\lambda/J)\boldsymbol{m}$ is solenoidal, as follows directly from comparison of (2.9) with (2.4) using (1.4) with $\boldsymbol{f} = (\lambda/J)\boldsymbol{m}$, $\boldsymbol{g} = \boldsymbol{v}$. The right-hand side of (2.11) then vanishes if S is a closed vorticity surface.

3. Discussion and conclusions

In this paper, the conservation of helicity has been extended, for steady fluid flow, to systems exhibiting material vorticity lines, the necessary and sufficient condition for which is given by (2.3) (Casey 1996). These fluid systems may be subject to non-conservative forces (such as viscous forces), but the latter are prohibited by (2.5) from drawing vorticity field lines away from vorticity surfaces, with the result that only advective processes determine the time-evolution of the vorticity field lines (Truesdell 1954). For the steady isochoric flow of a barotropic fluid governed by the Navier–Stokes equation under conservative body forces, (2.5) takes the form

$$\frac{\mathrm{D}\omega^0}{\mathrm{D}t}\frac{\boldsymbol{\omega}}{\omega^0} = \nu\nabla^2\boldsymbol{\omega}, \tag{3.1}$$

where ν is the kinematic viscosity. The physical significance of the logarithmic derivative of ω^0 on the left-hand side of (3.1) can be developed from an equation obtained after setting $\boldsymbol{f} = \boldsymbol{\omega}$, $\boldsymbol{g} = \boldsymbol{v}$ in the vector identity in (1.4):

$$\begin{aligned}\frac{\mathrm{d}\delta C}{\mathrm{d}t} &= \left(\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} - \boldsymbol{\omega}\cdot\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\omega}\boldsymbol{\nabla}\cdot\boldsymbol{v}\right)\cdot\delta\boldsymbol{S} \\ &= \boldsymbol{\nabla}\times(\boldsymbol{\omega}\times\boldsymbol{v})\cdot\delta\boldsymbol{S} \\ &= \frac{\mathrm{D}\omega^0}{\mathrm{D}t}\frac{\boldsymbol{\omega}}{\omega^0}\cdot\delta\boldsymbol{S} = \frac{\mathrm{D}\omega^0}{\mathrm{D}t}\frac{\delta C}{\omega^0},\end{aligned} \tag{3.2}$$

where

$$\delta C \equiv \boldsymbol{\omega}\cdot\delta\boldsymbol{S} \tag{3.3}$$

is the vortex strength of a material plane surface element $\delta\boldsymbol{S}$ located instantaneously at position $\boldsymbol{x}$, and a well-known result for the material derivative of $\delta\boldsymbol{S}$ has been used.

Equation (3.2) shows that the relative rate of change of the vortex strength of $\delta\boldsymbol{S}$ is the same as that of the Lagrangian vorticity magnitude ω^0, if (and only if) the field lines of $\boldsymbol{\omega}(\boldsymbol{x})$ are material. (This result also can be derived from Theorem I(a) in Casey & Naghdi (1991) using the invariance of the vortex strength of $\delta\boldsymbol{S}$ under Piola transformations (Marsden & Hughes 1994).) For steady viscous flows in which the curl of $\boldsymbol{\omega}\times\boldsymbol{v}$ is simultaneously equal to the right-hand sides of (2.5) and (3.1), the diffusive effect of viscosity impacts only the Eulerian vorticity magnitude ω, and the vortex strength of a material surface element changes only in sympathy with the Lagrangian vorticity magnitude ω^0. Physical examples in which (3.2) applies include steady isochoric viscous flows having stream lines that are geodesics on all stream surfaces (Marris 1969) and three-dimensional steady flows of a viscous fluid through an isotropic porous medium in which the hydraulic conductivity varies spatially (Bear 1988; Sposito 1997).

For the unsteady flows of an inviscid non-barotropic fluid, reference to its equation of motion (Mobbs 1981; Gaffet 1985) shows that

$$\nabla \times \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} \cdot \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = 0 \tag{3.4}$$

is the appropriate generalization of (1.6) for these fluid systems. The Casey & Naghdi (1991) theorem for unsteady fluid flows,

$$\nabla \times \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} = \frac{\mathrm{D}\omega^0}{\mathrm{D}t} \frac{\boldsymbol{\omega}}{\omega^0}, \tag{3.5}$$

generalizes (2.5) and implies, with (3.4), that the acceleration of an inviscid non-barotropic fluid is always perpendicular to its vorticity, if the field lines of the latter vector are material. Further physical insight follows from the Truesdell–Ertel–Beltrami vorticity-balance law (Truesdell 1954) for an arbitrary smooth fluid motion,

$$\frac{\mathrm{D}}{\mathrm{D}t}(J\boldsymbol{\omega} \cdot \nabla\theta) = J\boldsymbol{\omega} \cdot \nabla \frac{\mathrm{D}\theta}{\mathrm{D}t} + J\nabla \times \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} \cdot \nabla\theta, \tag{3.6}$$

where $\theta(\boldsymbol{x}, t)$ is any continuously differentiable tensor field. If $\theta(\boldsymbol{x}, t)$ is invariant under the fluid motion and the Casey & Naghdi (1991) theorem (3.5) applies, (3.6) reduces to the balance law:

$$\frac{\mathrm{D}}{\mathrm{D}t}(J\boldsymbol{\omega} \cdot \nabla\theta) = \frac{\mathrm{D}\omega^0}{\mathrm{D}t} \frac{J}{\omega^0} \boldsymbol{\omega} \cdot \nabla\theta, \tag{3.7}$$

which implies that the relative rate of change of $J\boldsymbol{\omega} \cdot \nabla\theta$ is the same as that of the magnitude of the Lagrangian vorticity. When $\theta(\boldsymbol{x}, t)$ is interpreted as specific entropy, $J\boldsymbol{\omega} \cdot \nabla\theta$ is proportional to the Ertel potential vorticity, and (3.7) is a balance law which generalizes those considered by Mobbs (1981), Gaffet (1985), and Salmon (1988). Conservation of the potential vorticity occurs if the fluid motion is also circulation-preserving (Truesdell 1954) or, equivalently, if ω^0 is constant (Casey & Naghdi 1991). Noting that the second step in (2.11),

$$\frac{\mathrm{d}\mathscr{H}'_S}{\mathrm{d}t} = \int_V (\lambda/J)\, \boldsymbol{m} \cdot \left(\nabla \tfrac{1}{2} v^2 + \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t}\right) \mathrm{d}V, \tag{3.8}$$

applies to unsteady fluid flows as well, one can conclude from the foregoing discussion that the second term on the right-hand side of (3.8) vanishes in the unsteady motion of an inviscid non-barotropic fluid and, by the line of reasoning applied originally to develop (2.11), so does the first term, if a closed vorticity surface envelopes the domain of volume V. Thus, when vorticity lines are material, $\mathscr{H}'_S$ as defined in (2.10) is an invariant of the unsteady motion of an inviscid non-barotropic fluid.

Geometrically speaking, (2.4) is a condition given long ago by Poincaré (1893) for the existence of a permanent family of surfaces on which stream lines and vorticity lines are inscribed in any steady flow. If moving a vorticity line along an intersecting stream line generates the same surface as is created by moving the stream line along the vorticity line, the difference on the left-hand side of (2.4) must be a zero vector in the two-dimensional space spanned by $\boldsymbol{m}$ and $\boldsymbol{v}$. This result is perhaps the simplest form of the well-known Frobenius condition for the existence of a surface to which $\boldsymbol{m}$ and

$\boldsymbol{v}$ are tangent everywhere (Berger & Gostiaux 1988). From a topological perspective, this surface is a smooth, connected, orientable two-dimensional manifold. For steady fluid flows having material vorticity lines, if the flow domain is compact, the manifold is (diffeomorphic to) a torus and is interspersed among a finite number of isolated stagnation points and separatrices. The torus underlies a topological classification of steady flows satisfying (2.9). The helicity invariant in (2.10) then may be used to augment this classification by indicating the complexity of linkage among the tori (Moffatt 1969, 1985, 1990; Moffatt & Ricca 1992).

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